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Seiberg-Witten type monopole equations on 8-manifolds with Spin(7) holonomy as minimizers of a quadratic action

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Abstract: We obtain an elliptic system of monopole equations on 8-manifolds with Spin(7) holonomy by minimizing an action involving negative spinors coupled to an abelian gauge field.

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1. Introduction

In Seiberg-Witten theory over a 4-manifold, a U(1) connection is coupled to a spinor field. The Seiberg-Witten equations are the minimizers of an action involving the curvature 2-form F and a positive Dirac spinor ϕ^+ (see eq. (2.7)),

$$I(A,\phi^+) = \int_M \left[|F|^2 + |\nabla\phi^+|^2 + \frac{1}{4}s|\phi^+|^2 + \frac{1}{4}|\phi^+|^4 q \right] \mathrm{dvol} \geq \int_M \mathrm{tr}F^2 \,, \qquad (1.1)$$

where s is the scalar curvature of the 4-manifold M and A and F are respectively the connection and curvature of a line bundle associated with the spin^c structure on M [1]. In the coupling of a Dirac spinor to the Yang-Mills action, the key is the Weitzenböck formula which relates the covariant derivatives $\nabla\phi^\pm$ to $D^\pm\phi^\pm$, where $D^\pm$ are Dirac operators (see eqs. (2.15)–(2.17)), compensating for the scalar curvature term and bringing in the coupling $(\rho^\pm(F)\phi^\pm, \phi^\pm)$.

In 4 dimensions, $\rho^+(F)$ is a 2×2 traceless, hermitean matrix such that $\rho^+(F^-)$ is identically zero. If $D^+\phi^+ = 0$, the action (1.1) can be written as

$$I(A,\phi^+) = \int_M F\wedge F + \int_M \left[2|F^+|^2 - (\rho^+(F^+)\phi^+, \phi^+) + \frac{1}{4}|\phi^+|^4 \right] \mathrm{dvol}\,. \qquad (1.2)$$

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Writing

$$\rho^+(F^+) = \begin{pmatrix} a & b+ic \\ b-ic & -a \end{pmatrix}, \qquad \phi^+ = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{1.3}$$

and using $2|F^+|^2 = |\rho^+(F^+)|^2$, the second integrand in eq. (1.2) is written as

$$\begin{aligned} 2|F^+|^2 - (\rho^+(F^+)\phi^+, \phi^+) + \frac{1}{4}|\phi^+|^4 = {} & [a^2+b^2+c^2] - \\ & - [a(\phi_1\bar{\phi}_1 - \phi_2\bar{\phi}_2) + b(\phi_1\bar{\phi}_2 + \phi_2\bar{\phi}_1) + \\ & \quad + ic(-\phi_1\bar{\phi}_2 + \phi_2\bar{\phi}_1)] + \\ & + \frac{1}{4}[\phi_1^2\bar{\phi}_1^2 + 2\phi_1\bar{\phi}_1\phi_2\bar{\phi}_2 + \phi_2^2\bar{\phi}_2^2] \\ = {} & \left[a - \frac{1}{2}(\phi_1\bar{\phi}_1 - \phi_2\bar{\phi}_2)\right]^2 + \left[b - \frac{1}{2}(\phi_1\bar{\phi}_2 + \phi_2\bar{\phi}_1)\right]^2 + \\ & + \left[c - \frac{i}{2}(-\phi_1\bar{\phi}_2 + \phi_2\bar{\phi}_1)\right]^2 . \end{aligned} \tag{1.4}$$

It can be seen that the vanishing of this term gives the second part of the Seiberg-Witten equations

$$D^+\phi^+ = 0\,, \qquad \rho^+(F^+) = [\phi^+(\bar{\phi}^+)^t]_o\,, \tag{1.5}$$

where the subscript o denotes the trace-free part of a matrix.

From eq. (1.4), it is clear that the term $\frac{1}{4}|\phi^+|^4$ is added just to complete the square. Choosing $\rho^+(F^+)$ as in eq. (1.5), reduces the action to its topological lower bound. In a sense, the spinors $\phi^\pm$ are chosen in such a way that the coupling is independent of the part of F of which it is expected to be free. This will be our idea in looking for Seiberg-Witten type monopole equations on 8-manifolds.

If one expects to get linear field equations as absolute minimizers of an action, the topological lower bound for the Yang-Mills functional would be an integral of the second Chern class c_2, which is proportional to $\mathrm{tr}(F^2)$. For an arbitrary 8-manifold, there is no canonical way of obtaining a topological invariant, either by integrating c_2 over a submanifold, or by multiplying c_2 with an appropriate (dimensionless) 4-form and integrating over M. Nevertheless, for eight manifolds with Spin(7) holonomy [2], there is a globally defined 4-form, called the *Bonan 4-form* [3, 4], which determines a topological lower bound for the Yang-Mills action

$$\int_M |F|^2 \mathrm{dvol} \geq \int_M F \wedge F \wedge \Phi\,. \tag{1.6}$$

The Bonan 4-form Φ belongs to the class of forms called *calibration forms* [5]. The 4-form Φ acting on 2-forms defines a "Φ-duality" by eq. (A.2) (see appendix). This determines a splitting of the curvature 2-form F into 7 and 21 dimensional subspaces as $F = F^{(7)} + F^{(21)}$. The requirement that $F^{(7)} = 0$ leads to a set of 7 equations, proposed originally by Corrigan et.al, [6]. Together with the Coulomb gauge condition, these equations from an elliptic system [7]. Gauge theories in higher dimensions in general were studied in [8]–[10]. The coupling of a spinor field to the gauge fields in 8-dimensions may follow different paths. A direct generalization of the Seiberg-Witten theory in 4-dimensions to 8-dimensions has

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led to trivial results [11]. In our previous paper [12], we defined a generalization of the monopole equations by interpreting the right hand sides of the Seiberg-Witten equations as a projection onto a subspace determined by the map ρ^+ and coupled positive spinors to the curvature. However, it was not possible to express these equations as absolute minimizers of an action. Another set of monopole equations in 8-dimensions is proposed by Gao and Tian [13] with the suggestion that it may provide the S-dual of non-abelian instantons [14].

In the present paper, we define monopole equations via a projection by the map ρ^-, thus coupling the curvature to negative spinors. This new set of monopole equations are absolute minimizers of an action, provided that the real and imaginary parts of the negative spinor belong to certain complementary subspaces of W^- determined by $\rho^-(\Phi)$.

2. Preliminaries

Notation. We define a hermitean inner product on $\mathrm{Hom}(C^n, C^m)$ by

$$(A, B) = \frac{1}{n}\,\mathrm{tr}(\bar{A}^t B)\,. \tag{2.1}$$

for A, B in $\mathrm{Hom}(C^n, C^m)$. In particular, if $X, Y \in C^n \simeq \mathrm{Hom}(C, C^n)$, and $A \in \mathrm{End}(C^n)$,

$$(AX, Y) = \mathrm{tr}(\bar{X}^t \bar{A}^t Y) = \mathrm{tr}(\bar{A}^t Y \bar{X}^t) = n(A, Y\bar{X}^t)\,. \tag{2.2}$$

Furthermore if A is skew-symmetric, then

$$(AX, X) = \frac{n}{2}(A, X\bar{X}^t - \bar{X}X^t)\,. \tag{2.3}$$

The pointwise norm of a real differential form $\omega \in \Lambda^*(T_p^*M)$ is defined as

$$|\omega|^2 = (\omega, \omega) = *(\omega \wedge *\omega)\,, \tag{2.4}$$

where $*$ denotes Hodge dual. We also note that if $F = i\omega$, where ω is real, then

$$(F, F) = (\omega, \omega), \quad F \wedge F = -\omega \wedge *\omega\,. \tag{2.5}$$

Spin^c structures. A detailed summary of the spin^c structures was given in [1]. We remind that a spin^c-structure on a $2n$-dimensional real inner-product space V is a pair (W, Γ), where Γ is a 2^n dimensional hermitean vector space and $\Gamma : V \to \mathrm{End}(W)$ is a linear map satisfying

$$\Gamma(v)^* = -\Gamma(v)\,, \qquad \Gamma(v)^2 = -|v|^2 \tag{2.6}$$

for $v \in V$. Globalization of this construction on each fiber to the tangent bundle of a $2n$-dimensional oriented manifold M will be possible if and only if the second Stiefel-Whitney class $w_2(M)$ has an integral lift. The spin^c structure on the manifold M will then be given by the map $\Gamma : TM \to \mathrm{End}(W)$, where W is now a 2^n-dimensional complex hermitean vector bundle on M. The $\pm i^n$ eigenspaces of $\Gamma(e_{2n}e_{2n-1}\dots e_1)$ determines a splitting

$$W = W^+ \oplus W^-\,, \tag{2.7}$$

and the elements of W^+ and W^- are called respectively "positive" and "negative" spinors.

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Restriction of $\Gamma(v)$ to W^- determines a linear map $\gamma : V \to \mathrm{Hom}(W^-, W^+)$ which satisfies $\gamma(v)^*\gamma(v) = |v|^2$ for all $v \in V$. The linear map can be recovered from γ via

$$\Gamma(v) = \begin{pmatrix} 0 & \gamma(v) \\ -\gamma(v)^* & 0 \end{pmatrix} . \tag{2.8}$$

To define a spin^c structure on V it is thus enough to give a linear map $\gamma : V \to \mathrm{Hom}(W^-, W^+)$ satisfying $\gamma(v)^*\gamma(v) = |v|^2$ for all $v \in V$.

In our case of an 8-manifold M with Spin(7)-holonomy, we will instantly construct such a map. But first we want to comment on the spinor bundle(s) of such a manifold. The holonomy condition means that the structure group of the frame bundle of the oriented manifold can be reduced from SO(8) to Spin(7) and, since Spin(7) is simply connected, the bundle group can be lifted to the double cover Spin(8) of SO(8). Thus, Spin(7)-holonomy implies that the manifold is Spin(8) and consequently, Spin^c(8).

Remark. The manifold M, being a Spin(8) manifold, possesses a real spinor bundle of dimension 16 and a complex spinor bundle of the same dimension, which is simply the complexification of the former. The above-mentioned complex spinor bundle W in the context of the Spin^c(8) structure on M is nothing else than the complex spinor bundle related to the Spin(8) structure on M, if the Spin^c(8) structure is the one canonically associated with the Spin(8) structure. Thus there are real spinor half-bundles $W_r^\pm$ with $W^\pm = W_r \otimes \mathbf{C}$, so that we can talk about real and imaginary parts of complex spinors. If on the other hand, Spin^c(8) is not the canonical one, our spinor bundle could be a twisted version of a real spinor bundle with a complex line bundle.

The extension of Γ to the even forms on M leads to a map $\rho : \Lambda^{2k}(M) \to \mathrm{End}(W)$ given by

$$\rho(e_i^* \wedge \cdots \wedge e_{2k}^*) = \Gamma(e_1) \cdots \Gamma(e_{2k}) . \tag{2.9}$$

With Γ as given in eq. (2.8), ρ will be block diagonal, and one can define the maps

$$\rho^\pm(\eta) = \rho(\eta) \mid_{W^\pm} , \tag{2.10}$$

for η in $\Lambda^{2k}(M)$.

The matrices $\gamma(e_i) = \gamma_i$ are characterized by

$$\gamma_1 = I , \qquad \gamma_j^2 + I = 0 , \qquad \gamma_j\gamma_k + \gamma_k\gamma_j = 0 , \qquad \text{for} \quad j \geq 2, j \neq k . \tag{2.11}$$

In 8-dimensions, the γ_j's are real, skew-symetric matrices and the set

$$\{\gamma_2, \gamma_3, \ldots, \gamma_8, \gamma_2\gamma_3, \ldots, \gamma_7\gamma_8\} \tag{2.12}$$

is orthonormal. The effects of the $\rho^\pm$ maps are given by

$$\begin{aligned} \rho^\pm(e_1^* \wedge e_j^*) &= \pm\gamma_j \, , \rho^\pm(e_j^* \wedge e_k^*) = \gamma_j\gamma_k \, , \\ \rho^\pm(e_1^* \wedge e_j^* \wedge e_k^* \wedge e_l^*) &= \pm\gamma_j\gamma_k\gamma_l \, , \qquad \rho^\pm(e_j^* \wedge e_k^* \wedge e_l^* \wedge e_m^*) = \gamma_j\gamma_k\gamma_l\gamma_m \, . \end{aligned} \tag{2.13}$$

The image of the volume form of M allows to define duality identifications. The γ_j matrices satisfy the relations $\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\gamma_7\gamma_8 = -I$ which allows the duality identifications

$$\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6 = \gamma_7\gamma_8 \qquad \gamma_2\gamma_3\gamma_4\gamma_5 = -\gamma_6\gamma_7\gamma_8 \, . \tag{2.14}$$

The Dirac operator and the Weitzenböck formula. A hermitean connection on W, compatible with the Levi-Civita connection of the manifold induces an imaginary valued connection on a certain associated line bundle called the *virtual line bundle* $L_\Gamma^{1/2}$. The corresponding connection 1-form is denoted by A and its curvature 2-form by $F \in \Omega^2(M, i\mathbf{R})$. The Dirac operators $D^{\pm}$ corresponding to A are the maps

$$\begin{aligned} D^+ &: C^\infty(M, W^+) \to C^\infty(M, W^-)\,, \\ D^- &: C^\infty(M, W^-) \to C^\infty(M, W^+) \end{aligned} \tag{2.15}$$

defined by

$$\begin{aligned} D^+(\phi^+) &= \sum_{i=1}^{2n} -\gamma^*(e_i)\nabla_{e_i}(\phi^+)\,, \\ D^-(\phi^-) &= \sum_{i=1}^{2n} \gamma(e_i)\nabla_{e_i}(\phi^-)\,, \end{aligned} \tag{2.16}$$

where $\{e_1, e_2, \ldots, e_{2n}\}$ is any local orthonormal frame.

The Weitzenböck formula is the key in writing the action integral for the Seiberg-Witten equations [5]. One has

$$\begin{aligned} D^-D^+\phi^+ &= \nabla^*\nabla\phi^+ + \frac{1}{4}s\phi^+ + \rho^+(F)\phi^+\,, \\ D^+D^-\phi^- &= \nabla^*\nabla\phi^- + \frac{1}{4}s\phi^- + \rho^-(F)\phi^-\,, \end{aligned} \tag{2.17}$$

where s is the scalar curvature of M and ∇^* is the L_2-adjoint of ∇, i.e.,

$$\int(\nabla^*\nabla\phi^{\pm}, \phi^{\pm})dvol = \int(\nabla\phi^{\pm}, \nabla\phi^{\pm})dvol\,. \tag{2.18}$$

Taking the inner product of eq. (2.17) with $\phi^{\pm}$ and integrating over M, we obtain

$$\int_M |D^{\pm}\phi^{\pm}|^2 dvol = \int_M \left[|\nabla\phi^{\pm}|^2 + \frac{1}{4}s|\phi^{\pm}|^2 + (\rho^{\pm}(F)\phi^{\pm}, \phi^{\pm})\right]. \tag{2.19}$$

Hence if $D^-\phi^- = 0$, we have

$$\int[|\nabla\phi^-|^2 + \frac{1}{4}s|\phi^-|^2]dvol = -\int(\rho^-(F)\phi^-, \phi^-)dvol\,. \tag{2.20}$$

The topological lower bound. The expression of the Bonan 4-form Φ in an orthonormal basis and its eigenvectors are given in appendix. The eigenvectors $\{\omega_{1j}\}$ and $\{\omega_{jk}\}$ for $j,k = 2,\ldots,8$ are a basis for local sections of $\Lambda^2(M)$, and the curvature 2-form F has the splitting $F = F^{(7)} + F^{(21)}$, where $F^{(7)} = \sum ia_j\omega_{1j}$, $F^{(21)} = \sum ia_{kl}\omega_{kl}$. It can be seen that

$$F^{(7)} \wedge F^{(21)} \wedge \Phi = 0\,, \tag{2.21}$$

hence

$$F \wedge F \wedge \Phi = F^{(7)} \wedge F^{(7)} \wedge \Phi + F^{(21)} \wedge F^{(21)} \wedge \Phi\,. \tag{2.22}$$

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As F is pure imaginary,

$$F^{(7)} \wedge F^{(7)} \wedge \Phi = -3 * (F^{(7)}, F^{(7)}) ,$$
$$F^{(21)} \wedge F^{(21)} \wedge \Phi = *(F^{(21)}, F^{(21)}) . \tag{2.23}$$

Hence

$$(F^{(21)}, F^{(21)}) = 3(F^{(7)}, F^{(7)}) + *(F \wedge F \wedge \Phi) . \tag{2.24}$$

It follows that

$$|F|^2 = 4|F^{(7)}|^2 + \frac{2}{3} * (F \wedge F \wedge \Phi) . \tag{2.25}$$

Note that we can also write

$$|F|^2 = \frac{4}{3}|F^{(21)}|^2 - \frac{2}{3} * (F \wedge F \wedge \Phi) . \tag{2.26}$$

3. The monopole equations

We have previously given [12] an elliptic set of monopole equations using a projection with $\rho^+(F^{(7)})$, where we denoted $F^{(7)}$ as F^+, and ω_{1j} by f_{j-1}.

The same formula ([12, eq. (26)]) can be applied as well using a projection with ρ^-. Hence we can write

$$\rho^\pm(F^{(7)}) = \sum_{j=2}^{8} \langle \rho^\pm(\omega_{1j}), \phi^\pm(\phi^\pm)^* \rangle |\rho^\pm(\omega_{1j})|^{-2} \rho^\pm(\omega_{1j}) , \tag{3.1}$$

where ω_{1j}'s are an orthonormal basis for the subspace $F^{(7)}$. We may choose the ω_{1j}'s as in (A.3). With respect to this basis we can write

$$F^{(7)} = \sum_{j=2}^{8} i a_{1j} \omega_{1j} , \tag{3.2}$$

then

$$\rho^\pm(F^{(7)}) = \sum_{j=2}^{8} i a_{1j} \rho^\pm(\omega_{1j}) , \tag{3.3}$$

and the monopole equations are

$$i a_{1j} = |\rho^\pm(\omega_{1j})|^{-2} \langle \rho^\pm(\omega_{1j}), \phi^\pm(\bar{\phi}^\pm)^t \rangle , \qquad j = 2, \ldots, 8 . \tag{3.4}$$

We now discuss the effects of the maps $\rho^\pm$. In the equations that follow, the ω_{1j}'s are determined via the Spin(7) invariant Bonan 4-form. The first terms at the right hand sides of the equations are the the expressions of $\rho^\pm(\omega_{1j})$'s for an arbitrary spin^c structure, while the second set of terms are the expressions corresponding to the spin^c structure given by $\gamma_j = \omega_{1j}$. Here we remind that there is an abuse of notation, as ω_{1j} denotes the skew

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symmetric matrix corresponding to the 2-form. Similarly in eqs. (3.5) and (3.6), e_{jk}'s are skew symmetric matrices. We use the notation $\gamma_{jk} = \gamma_j \gamma_k$. Then

$$
\begin{aligned}
\rho^+(\omega_{12}) &= \gamma_5 + \gamma_{26} + \gamma_{37} + \gamma_{48} = 2[e_{13} - e_{24} - e_{57} - e_{68}]\,, \\
\rho^+(\omega_{13}) &= \gamma_2 + \gamma_{34} - \gamma_{56} - \gamma_{78} = 2[e_{15} - e_{26} + e_{37} + e_{48}]\,, \\
\rho^+(\omega_{14}) &= \gamma_6 - \gamma_{25} - \gamma_{38} + \gamma_{47} = 2[e_{17} + e_{28} - e_{35} + e_{46}]\,, \\
\rho^+(\omega_{15}) &= \gamma_3 - \gamma_{24} - \gamma_{57} + \gamma_{68} = 2[e_{12} + e_{34} + e_{56} - e_{78}]\,, \\
\rho^+(\omega_{16}) &= \gamma_7 + \gamma_{28} - \gamma_{35} - \gamma_{46} = 2[e_{14} + e_{23} - e_{58} + e_{67}]\,, \\
\rho^+(\omega_{17}) &= \gamma_4 + \gamma_{23} - \gamma_{58} - \gamma_{67} = 2[-e_{16} - e_{25} - e_{38} + e_{47}]\,, \\
\rho^+(\omega_{18}) &= \gamma_8 - \gamma_{27} + \gamma_{36} - \gamma_{45} = 2[e_{18} - e_{27} - e_{36} - e_{45}]\,,
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
\rho^-(\omega_{12}) &= -\gamma_5 + \gamma_{26} + \gamma_{37} + \gamma_{48} = -4e_{68}\,, \\
\rho^-(\omega_{13}) &= -\gamma_2 + \gamma_{34} - \gamma_{56} - \gamma_{78} = -4e_{26}\,, \\
\rho^-(\omega_{14}) &= -\gamma_6 - \gamma_{25} - \gamma_{38} + \gamma_{47} = 4e_{46}\,, \\
\rho^-(\omega_{15}) &= -\gamma_3 - \gamma_{24} - \gamma_{57} + \gamma_{68} = 4e_{56}\,, \\
\rho^-(\omega_{16}) &= -\gamma_7 + \gamma_{28} - \gamma_{35} - \gamma_{46} = 4e_{67}\,, \\
\rho^-(\omega_{17}) &= -\gamma_4 + \gamma_{23} - \gamma_{58} - \gamma_{67} = -4e_{16}\,, \\
\rho^-(\omega_{18}) &= -\gamma_8 - \gamma_{27} + \gamma_{36} - \gamma_{45} = -4e_{36}\,.
\end{aligned} \tag{3.6}
$$

We shall not give the explicit expression of the $\rho^-(\omega_{jk})$'s, but we indicate the method of computation. As an example, we compute $\rho^-(\omega)$ for $\omega = e_{12} + ae_{34} + be_{56} + ce_{78}$. For $1 + a - b - c = 0$, ω belongs to the 21 dimensional subspace. We have chosen the labeling of the ω_{jk}'s so that if $\gamma_j = \omega_{1j}$, then the matrix $\gamma_{jk} = \gamma_j \gamma_k$ is the skew symmetric matrix corresponding to the 2-form ω_{jk}. Then using eqs. (A.3) and (A.4) we have

$$
\begin{aligned}
\rho^-(\omega) &= -\gamma_2 + a\gamma_{34} + b\gamma_{56} + c\gamma_{78} \\
&= -(e_{15} + e_{26} + e_{37} + e_{48}) - a(e_{15} + e_{26} - e_{37} - e_{48}) - b(e_{15} - e_{26} + e_{37} - e_{48}) - \\
&\quad - c(e_{15} - e_{26} - e_{37} + e_{48}) \\
&= -(1 + a + b + c)e_{15} + (-1 - a + b + c)e_{26} + (-1 + a - b + c)e_{37} + \\
&\quad + (-1 + a + b - c)e_{48}\,.
\end{aligned} \tag{3.7}
$$

Note that for ω in the 21 dimensional subspace, the coefficient of e_{26} is zero. With similar computations it can be seen that the coefficients of e_{j6} vanish for each j, whenever ω belongs to the 21 dimensional space. It follows that the 6th row and the 6th column of $\rho^-(F^{(21)})$ are zero. This will be the key fact in constructing the action integral.

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For the specific spin^c structure above, our monopole equations corresponding to the coupling to a positive spinor [12] are

$$\begin{aligned}
4a_{12} &= F_{15}+F_{26}+F_{37}+F_{48} \\
&= \frac{1}{4}(\phi_1\bar{\phi}_3-\phi_3\bar{\phi}_1-\phi_2\bar{\phi}_4+\phi_4\bar{\phi}_2-\phi_5\bar{\phi}_7+\phi_7\bar{\phi}_5-\phi_6\bar{\phi}_8+\phi_8\bar{\phi}_6)\,, \\
4a_{13} &= F_{12}+F_{34}-F_{56}-F_{78} \\
&= \frac{1}{4}(\phi_1\bar{\phi}_5-\phi_5\bar{\phi}_1-\phi_2\bar{\phi}_6+\phi_6\bar{\phi}_2+\phi_3\bar{\phi}_7-\phi_7\bar{\phi}_3+\phi_4\bar{\phi}_8-\phi_8\bar{\phi}_4)\,, \\
4a_{14} &= F_{16}-F_{25}-F_{38}+F_{47} \\
&= \frac{1}{4}(\phi_1\bar{\phi}_7-\phi_7\bar{\phi}_1+\phi_2\bar{\phi}_8-\phi_8\bar{\phi}_2-\phi_3\bar{\phi}_5+\phi_5\bar{\phi}_3+\phi_4\bar{\phi}_6-\phi_6\bar{\phi}_4)\,, \\
4a_{15} &= F_{13}-F_{24}-F_{57}+F_{68} \\
&= \frac{1}{4}(\phi_1\bar{\phi}_2-\phi_2\bar{\phi}_1+\phi_3\bar{\phi}_4-\phi_4\bar{\phi}_3+\phi_5\bar{\phi}_6-\phi_6\bar{\phi}_5-\phi_7\bar{\phi}_8+\phi_8\bar{\phi}_7)\,, \\
4a_{16} &= F_{17}+F_{28}-F_{35}-F_{46} \\
&= \frac{1}{4}(\phi_1\bar{\phi}_4-\phi_4\bar{\phi}_1+\phi_2\bar{\phi}_3-\phi_3\bar{\phi}_2-\phi_5\bar{\phi}_8+\phi_8\bar{\phi}_5+\phi_6\bar{\phi}_7-\phi_7\bar{\phi}_6)\,, \\
4a_{17} &= F_{14}+F_{23}-F_{58}-F_{67} \\
&= \frac{1}{4}(-\phi_1\bar{\phi}_6+\phi_6\bar{\phi}_1-\phi_2\bar{\phi}_5+\phi_5\bar{\phi}_2-\phi_3\bar{\phi}_8+\phi_8\bar{\phi}_3+\phi_4\bar{\phi}_7-\phi_7\bar{\phi}_4)\,, \\
4a_{18} &= F_{18}-F_{27}+F_{36}-F_{45} \\
&= \frac{1}{4}(\phi_1\bar{\phi}_8-\phi_8\bar{\phi}_1-\phi_2\bar{\phi}_7+\phi_7\bar{\phi}_2-\phi_3\bar{\phi}_6+\phi_6\bar{\phi}_3-\phi_4\bar{\phi}_5+\phi_5\bar{\phi}_4)\,,
\end{aligned} \tag{3.8}$$

while for the coupling to a negative spinor here we give the monopole equations

$$\begin{aligned}
4a_{12} &= F_{15}+F_{26}+F_{37}+F_{48} = \frac{1}{2}(-\phi_6\bar{\phi}_8+\phi_8\bar{\phi}_6)\,, \\
4a_{13} &= F_{12}+F_{34}-F_{56}-F_{78} = \frac{1}{2}(-\phi_2\bar{\phi}_6+\phi_6\bar{\phi}_2)\,, \\
4a_{14} &= F_{16}-F_{25}-F_{38}+F_{47} = \frac{1}{2}(\phi_4\bar{\phi}_6-\phi_6\bar{\phi}_4)\,, \\
4a_{15} &= F_{13}-F_{24}-F_{57}+F_{68} = \frac{1}{2}(\phi_5\bar{\phi}_6-\phi_6\bar{\phi}_5)\,, \\
4a_{16} &= F_{17}+F_{28}-F_{35}-F_{46} = \frac{1}{2}(\phi_6\bar{\phi}_7-\phi_7\bar{\phi}_6)\,, \\
4a_{17} &= F_{14}+F_{23}-F_{58}-F_{67} = \frac{1}{2}(-\phi_1\bar{\phi}_6+\phi_6\bar{\phi}_1)\,, \\
4a_{18} &= F_{18}-F_{27}+F_{36}-F_{45} = \frac{1}{2}(-\phi_3\bar{\phi}_6+\phi_6\bar{\phi}_3)\,.
\end{aligned} \tag{3.9}$$

4. The action integral

We prove now that the monopole equations (3.9) for $F^{(7)}$ and ϕ^- can be derived from an action principle. However, this is not possible for any ϕ^-, but for a special one given by eq. (4.1) below. The subtle point here is that as γ_i's are real, skew-symmetric matrices, U can always be chosen locally as real.

We recall that M is an 8-manifold with Spin(7) holonomy, with scalar curvature s and Bonan 4-form Φ. Given any spin^c structure (Γ, W), the subspace W^- and the map ρ^- are

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defined by (2.7) and (2.10), A is a U(1) connection on the "virtual line bundle" $L_\Gamma^{1/2}$ (as in section 2) and F is the curvature of A. The Bonan 4-form determines an orthogonal direct sum decomposition of the 2-form F by the map A.2, and $F^{(7)}$ and $F^{(21)}$ are the projections of F onto these subspaces.

Proposition 1. *Let M, F and Φ be as above and define a section ϕ of W^- by*

$$\phi = (1-P)U + iPU\,, \tag{4.1}$$

where U is a real section of W^- and

$$P = \frac{1}{8} - \frac{1}{16}\rho^-(\Phi)\,. \tag{4.2}$$

Then the monopole equations

$$D^-\phi = 0\,, \qquad \rho^-(F^{(7)}) = \frac{1}{2}(\phi\bar{\phi}^t - \bar{\phi}\phi^t)\,, \qquad \mathrm{div}A = 0 \tag{4.3}$$

are absolute minimizers of the action

$$I(A,U) = \int_M \left[|F|^2 + |\nabla\phi|^2 + \frac{1}{4}s|\phi|^2 + |\phi\bar{\phi}^t - \bar{\phi}\phi^t|^2\right] dvol \tag{4.4}$$

and $I(A,U) \geq \frac{2}{3}\int_M F\wedge F\wedge\Phi$.

Proof. We first use eqs. (2.25) and (2.20) to express the action as

$$I(A,U) = \frac{2}{3}\int_M F\wedge F\wedge\Phi + \int_M \left[4|F^{(7)}|^2 - (\rho^-(F)\phi,\phi) + \frac{1}{4}s|\phi|^2 + |\phi\bar{\phi}^t - \bar{\phi}\phi^t|^2\right] dvol\,. \tag{4.5}$$

Next we show that $|F^{(7)}|^2 = |\rho^-(F^{(7)})|^2$. For this note that $\rho^-(\omega_{1j})$ is of the form $-\gamma_i + \gamma_j\gamma_k + \gamma_m\gamma_n + \gamma_p\gamma_q$ (see eqs. (3.6)) with distinct set of indices. The orthonormality of the set (2.12) implies that $|\rho^-(\omega_{1j})|^2 = 4$. On the other hand from (A.3), $|\omega_{1j}|^2 = 4$, hence the ρ^- map is an isometry. It follows that

$$I(A,U) = \frac{2}{3}\int_M F\wedge F\wedge\Phi + \int_M \left[4|\rho^-(F^{(7)})|^2 - (\rho^-(F)\phi,\phi) + \frac{1}{4}s|\phi|^2 + |\phi\bar{\phi}^t - \bar{\phi}\phi^t|^2\right] dvol\,. \tag{4.6}$$

The main difficulty is to show that $(\rho^-(F^{(21)})\phi,\phi) = 0$ for ϕ given by eq. (4.1). For this we first prove the following Lemma.

Lemma 1. *Let P be as in eq. (4.2). Then*

$$\rho^-(F^{(21)})P = 0\,, \qquad \rho^-(F^{(7)})P + P\rho^-(F^{(7)}) = \rho^-(F^{(7)})\,. \tag{4.7}$$

Proof of Lemma. Let $Q = -\frac{1}{2}\rho^-(\Phi)$. For Φ given by eq. (A.1), Q is the symmetric matrix given by

$$Q = \gamma_2\gamma_3\gamma_4 - \gamma_2\gamma_5\gamma_6 - \gamma_2\gamma_7\gamma_8 - \gamma_3\gamma_5\gamma_7 + \gamma_3\gamma_6\gamma_8 - \gamma_4\gamma_5\gamma_8 - \gamma_4\gamma_6\gamma_7\,. \tag{4.8}$$

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Using the duality relations (2.14), it can be seen that $Q^2 = 7 + 6Q$, hence the minimal polynomial of Q is

$$(Q - 7I)(Q + I) = 0\,. \tag{4.9}$$

Let the dimension of the eigenspaces corresponding to the eigenvalues 7 and -1 be m and $8 - m$. The orthogonality of the set (2.12) implies that Q is traceless. On the other hand, $trQ = 7m + (-1)(8 - m) = 0$, hence $m = 1$. If we set $P = \frac{1}{8}(Q + I)$, then P has rank 1 and

$$P^2 = P\,, \tag{4.10}$$

hence P is a projection onto a 1-dimensional subspace of W^-. To prove eq. (4.7) we need to compute the relevant expressions for each basis element. We give the details of the computation for

$$\omega = e_{12} + ae_{34} + be_{56} + ce_{78} \tag{4.11}$$

which will prove the result for ω_{13}, ω_{24}, ω_{68} and ω_{75}. We rewrite $(Q + I)$ four times with appropriate duality identifications and multiply with ω. We use below the notation $\gamma_{ijk} = \gamma_i\gamma_j\gamma_k$.

$$\begin{aligned}(Q+I)\omega \;=\;& (I + \gamma_{234} - \gamma_{256} - \gamma_{278} + \gamma_{2468} - \gamma_{2457} + \gamma_{2367} + \gamma_{2358})(-\gamma_2)\times\\ &\times(I + \gamma_{234} + \gamma_{3478} + \gamma_{3456} - \gamma_{357} + \gamma_{368} - \gamma_{458} - \gamma_{467})(a\gamma_{34})\times\\ &\times(I - \gamma_{5678} - \gamma_{256} + \gamma_{3456} - \gamma_{357} + \gamma_{368} - \gamma_{458} - \gamma_{467})(a\gamma_{56})\times\\ &\times(I - \gamma_{5678} + \gamma_{3478} - \gamma_{278} - \gamma_{357} + \gamma_{368} - \gamma_{458} - \gamma_{467})(a\gamma_{78})\\ =\;& (1 + a - b - c)(-\gamma_2 + \gamma_{34} - \gamma_{56} - \gamma_{78} - \gamma_{468} + \gamma_{457} - \gamma_{367} + \gamma_{358})\,.\end{aligned} \tag{4.12}$$

If ω belongs to the 21 dimensional subspace then $1+a-b-c = 0$, and hence $(Q+I)\rho^-(\omega) = 0$. On the other hand if ω belongs to the 7-dimensional subspace, $(Q + I)\rho^-(\omega) \neq 0$, but it can be seen that

$$(Q + I)\rho^-(\omega) + \rho^-(\omega)(Q + I) = 2(1 + a - b - c)(-\gamma_2 + \gamma_{34} + \gamma_{56} + \gamma_{78}) \tag{4.13}$$

and as $a = 1$, $b = c = -1$, it follows that $(Q + I)\rho^-(\omega) + \rho^-(\omega)(Q + I) = 8\rho^-(\omega)$ and we obtain eq. (4.7).

Remark. For the specific choice of the spin^c structure corresponding to $\gamma_j = \omega_{1j}$, P is the diagonal matrix with the only nonzero entry, $P_{66} = 1$. From the remark after eq. (3.7) we see that the 6th row and the 6th column of $\rho^-(\omega_{jk})$'s are zero while in $\rho^-(\omega_{1j})$'s non-zero elements are only in the 6th row and the 6th column.

We now continue with the proof of proposition 1. First note that as F is pure imaginary and the representation is real, $\rho^-(F)$ is hermitean but skew-symmetric. Now using $\rho^-(F^{(21)})P = 0$, we have

$$(\rho^-(F^{(21)})\phi, \phi) = (\rho^-(F^{(21)})U, (1 - P)U + iPU) = (U, \rho^-(F^{(21)})U) = 0\,, \tag{4.14}$$

where the second equality follows from hermiticity and the third from skew-symmetry.

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Then from eq. (2.3),

$$(\rho^-(F^{(7)})\phi,\phi) = 4(\rho^-(F^{(7)}),\phi\bar{\phi}^t - \bar{\phi}\phi^t) \tag{4.15}$$

and eq. (4.6) reduces to

$$I(A,U) = \frac{2}{3}\int_M F\wedge F\wedge\Phi + \int_M \left[4|\rho^-(F^{(7)})|^2 - 4(\rho-(F^{(7)}),\phi\bar{\phi}^t - \bar{\phi}\phi^t) + |\phi\bar{\phi}^t - \bar{\phi}\phi^t|^2\right] dvol \tag{4.16}$$

which can be written as

$$I(A,U) = \frac{2}{3}\int_M F\wedge F\wedge\Phi + \int_M |2\rho^-(F^{(7)}) - (\phi\bar{\phi}^t - \bar{\phi}\phi^t)|^2 dvol \tag{4.17}$$

which gives the monopole equations.

As the left hand side of the matrix equation $\rho^-(F^{(7)}) = \frac{1}{2}(\phi\bar{\phi}^t - \bar{\phi}\phi^t)$ belongs to a 7-dimensional subspace, we need to check its compatibility. Given any skew-symmetric matrix A, there is an orthogonal decomposition

$$A = (PA + AP) + (I-P)A(I-P)\,. \tag{4.18}$$

The orthogonality of the summands can be seen from

$$\mathrm{tr}((PA+AP)(I-P)A(I-P)) = \mathrm{tr}(PA(I-P)A(I-P)) = \mathrm{tr}(A(I-P)A(I-P)P) = 0\,. \tag{4.19}$$

From eq. (4.7), $\rho^-(F^{(7)}) = P\rho^-(F^{(7)}) + \rho^-(F^{(7)})P$. Hence we need to show that the right hand side satisfies the same relation. It can be seen that

$$\frac{1}{2}(\phi\bar{\phi}^t - \bar{\phi}\phi^t) = \frac{1}{2}[-i(I-P)UU^tP + iPUU^t(I-P)] = i[-UU^tP + PUU^t]\,. \tag{4.20}$$

But

$$P[PUU^t - UU^tP] + [PUU^t - UU^tP]P = PUU^t - UU^tP\,, \tag{4.21}$$

so both sides of the equation belong to the same linear subspace. This completes the proof of proposition 1.

For $\gamma_j = \omega_{1j}$, the explicit form of the monopole equations can be obtained by putting

$$\bar{\phi}_6 = -\phi_6\,, \qquad \bar{\phi}_j = \phi_j\,, \qquad \text{for} \quad j \neq 6 \tag{4.22}$$

in eqs. (3.9).

Finally, we shall show that the new set of equations are elliptic. In a system of n first order partial differential equations in the independent variables $x_1,\ldots,x_k$ and the dependent variables $u_1,\ldots,u_n$, the replacement of $\frac{\partial u_i}{\partial x_j}$ by $\xi_j u_i$, where ξ_j's are indeterminates, one obtains (see e.g. [15]) a linear homogeneous system of equations for the u_i's. The characteristic determinant of the system is the determinant of the coeficient matrix of the corresponding linear system. The system of partial differential equations is called elliptic, if the characteristic determinant has no real roots. The ellipticity property of a system is independent of its inhomogeneous part.

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Proposition 2. *The system of equations (4.3) is elliptic.*

Proof. It is known that the system of equations $F^{(7)} = 0$ together with the Coulomb gauge condition $div(A) = 0$ are elliptic [9]. If $F^{(7)} = \sum_{j=2}^{8} i a_j \omega_{1j}$, as $\rho^-(\omega_{1j})$'s are linearly independent, $\rho^-(F^{(7)}) = 0$ if and only if $F^{(7)} = 0$. Thus the inhomogeneous system $\rho^-(F^{(7)}) = \frac{1}{2}(\phi\bar{\phi}^t - \bar{\phi}\phi^t)$ together with the Coulomb gauge condition is elliptic. To prove the ellipticity of the system $D^-\phi = 0$, let $\phi = SU$ and K be the characterisitic determinant of the system $D^-U = 0$. It can be seen that the characteristic determinant of the system $D^-(SU) = 0$ is $\tilde{K} = K det(S)$. Hence we need to show that S is nonsingular. Using the minimal polynomial of Q it can be checked that the minimal polynomial of S is

$$S^2 - (1+i)S + i \tag{4.23}$$

which shows that it is nonsingular and completes the proof.

A. Eigenvectors of the Bonan 4-form

We recall that, in an orthonormal basis $\{e_i\}$, $i = 1, \ldots, 8$, the Bonan 4-form Φ can be written as

$$\begin{aligned} \Phi = [&e_{1234} - e_{1256} - e_{1278} - e_{1357} + e_{1368} - e_{1458} - e_{1467} + \\ &+ e_{5678} - e_{3478} - e_{3456} - e_{2468} + e_{2457} - e_{2367} - e_{2358}] \,, \end{aligned} \tag{A.1}$$

where $e_{ijkl} = e_i^* \wedge e_j^* \wedge e_k^* \wedge e_l^*$. The form Φ is Spin(7) invariant, hence it can be extended to the manifold. Φ is self-dual and it defines a linear map on 2-forms as

$$\omega \to *(\Phi \wedge \omega) \,, \tag{A.2}$$

with eigenvalues 3 and -1. This map has 7 and 21 dimensional eigenspaces and a basis consisting of the corresponding eigenvectors are given below.

Eigenvectors corresponding to the eigenvalue 3**:**

$$\begin{aligned} \omega_{12} &= e_{15} + e_{26} + e_{37} + e_{48} \\ \omega_{13} &= e_{12} + e_{34} - e_{56} - e_{78} \\ \omega_{14} &= e_{16} - e_{25} - e_{38} + e_{47} \\ \omega_{15} &= e_{13} - e_{24} - e_{57} + e_{68} \\ \omega_{16} &= e_{17} + e_{28} - e_{35} - e_{46} \\ \omega_{17} &= e_{14} + e_{23} - e_{58} - e_{67} \\ \omega_{18} &= e_{18} - e_{27} + e_{36} - e_{45} \,. \end{aligned} \tag{A.3}$$

The labeling of the eigenvectors corresponding to the eigenvalue -1 is done as follows. By abuse of notation, let ω_{1j} be the skew-symmetric matrix corresponding to the 2-form ω_{1j}. Then ω_{jk} is the 2-form corresponding to the matrix $\omega_{1j}\omega_{1k}$.

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Eigenvectors corresponding to the eigenvalue -1:

$$
\begin{aligned}
\omega_{87} &= e_{15} - e_{26} - e_{37} + e_{48} \\
\omega_{68} &= e_{12} - e_{34} + e_{56} - e_{78} \\
\omega_{67} &= e_{16} + e_{25} + e_{38} + e_{47} \\
\omega_{84} &= e_{13} + e_{24} + e_{57} + e_{68} \\
\omega_{74} &= e_{17} - e_{28} + e_{35} - e_{46} \\
\omega_{46} &= e_{14} - e_{23} + e_{58} - e_{67} \\
\omega_{72} &= e_{18} + e_{27} - e_{36} - e_{45} \\
\\
\omega_{43} &= e_{15} + e_{26} - e_{37} - e_{48} \\
\omega_{75} &= e_{12} + e_{34} + e_{56} + e_{78} \\
\omega_{32} &= e_{16} - e_{25} + e_{38} - e_{47} \\
\omega_{37} &= e_{13} - e_{24} + e_{57} - e_{68} \\
\omega_{83} &= e_{17} + e_{28} + e_{35} + e_{46} \\
\omega_{53} &= e_{14} + e_{23} + e_{58} + e_{67} \\
\omega_{36} &= e_{18} - e_{27} - e_{36} + e_{45} \\
\\
\omega_{65} &= e_{15} - e_{26} + e_{37} - e_{48} \\
\omega_{24} &= e_{12} - e_{34} - e_{56} + e_{78} \\
\omega_{58} &= e_{16} + e_{25} - e_{38} - e_{47} \\
\omega_{26} &= e_{13} + e_{24} - e_{57} - e_{68} \\
\omega_{52} &= e_{17} - e_{28} - e_{35} + e_{46} \\
\omega_{28} &= e_{14} - e_{23} - e_{58} + e_{67} \\
\omega_{45} &= e_{18} + e_{27} + e_{36} + e_{45} \,.
\end{aligned}
\tag{A.4}
$$

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